1 Introduction

TBD

2 Preliminaries

Definition 1 (bC^α space)**.** *bC^α is the space of bounded α-Holder functions normed by*

$$||f||_{bC^\alpha} = \sup_x |f(x)| + \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

Definition 2 (d_{bC^α} on $\mathcal{P}(\mathbb{R}^N)$)**.** *We define on $\mathcal{P}(\mathbb{R}^N)$ the distance*

$$d_{bC^\alpha}(\mu,\nu) = \sup_{||f||_{bC^\alpha}\leq 1} \int f(x)\left(\mu(dx)-\nu(dx)\right).$$

Definition 3 (MKV SDE)**.** *A MKV SDE is an SDE of the type*

$$dX_t = B_t(X_t,[X_t])dt + \Sigma_t(X_t,[X_t])dW_t,$$

where W_t is a N-dimensional Brownian Motion, the coefficients are functions $B:\Omega\times[0,T]\times\mathbb{R}^N\times\mathcal{P}(\mathbb{R}^N)\to\mathbb{R}^N$ and $\Sigma:\Omega\times[0,T]\times\mathbb{R}^N\times\mathcal{P}(\mathbb{R}^N)\to\mathbb{R}^{N\times N}$ and lastly $[X_t]$ is the distribution of X_t.

Assumption 4 (Structural Assumption)**.** *Consider a MKV SDE where the coefficients are defined as*

$$B(t,x,\mu) = \int_{\mathbb{R}^N} b(t,x,y)\mu(dy), \qquad \Sigma(t,x,\mu) = \int_{\mathbb{R}^N} \sigma(t,x,y)\mu(dy),$$

where b and σ are deterministic functions of $b:[0,T]\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$ and $\sigma:[0,T]\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^{N\times N}$. In particular the coefficients are deterministic and the dependence on the distribution is fixed and integral-like.

Assumption 5 (Non degeneracy)**.** *The diffusion coefficient Σ defines a uniformly parabolic operator, precisely*

$$\exists\lambda>0 \ s.t.\ \frac{1}{\lambda}|v|^2 \leq \langle\Sigma^*(t,x,\mu)\Sigma(t,x,\mu)v,v\rangle \leq \lambda|v|^2,\ \ \forall v\in\mathbb{R}^N,\forall(t,x,\mu)\in[0,T]\times\mathbb{R}^N\times\mathcal{P}(\mathbb{R}^N),$$

a sufficient condition for this to occur under assupmtion 4 is if σ is a uniformly positive definite matrix.

Assumption 6 (Regularity)**.** *Assume assumption 4. The coefficients b and σ are functions in $L^\infty([0,T],bC^\alpha)$. Explicitly*

$$\begin{aligned}\exists C>0,\ s.t.\ \forall(x_1,x_2,y_1,y_2)\in\mathbb{R}^{4N},\ t-\text{a.s.},\qquad |b(t,x,y)|+|\sigma(t,x,y)|\leq C,\\ \text{and } s.t.\ |b(t,x_1,y_1)-b(t,x_2,y_2)|+|\sigma(t,x_1,y_1)-\sigma(t,x_2,y_2)|\leq C\left(|x_1-x_2|^\alpha+|y_1-y_2|^\alpha\right).\end{aligned}$$

3 Results

Theorem 7 (Non-degenerate case)**.** *Consider a MKV SDE under assumptions 4, 5 and 6 and initial distribution $\mu_0\in\mathcal{P}(\mathbb{R}^N)$; then we have weak existance and uniqueness of the solution of the MKV SDE.*

4 Proofs

Proof. of Theorem 7.

The core of the proof is a contraction argument on the space of the flows of marginals, indeed if it is possible to show that the flow of marginals is unique it can be fixed in the MKV SDE to reduce it to a classical SDE, at this point weak well posedness results for classical SDEs will conclude the proof.

To begin if we consider a flow of marginals $(\mu_t)_{t\in[0,T]}\in C([0,T],\mathcal{P}(\mathbb{R}^N))$ and fix it inside the coefficients we are able to construct "linearized" coefficients and a classical SDE

$$\begin{aligned}B^\mu(t,x) = B(t,x,\mu_t), \qquad \Sigma^\mu(t,x) = \Sigma(t,x,\mu_t),\\ dX^\mu_t = B^\mu(t,X^\mu_t)dt + \Sigma^\mu(t,X^\mu_t)dW_t \qquad X^\mu_0\sim\mu_0.\end{aligned}$$

We first notice that the function $\mu\to B(t,x,\mu)\in C(\mathcal{P}(\mathbb{R}^N),\mathbb{R}^N)$ since

$$|B(t,x,\mu)-B(t,x,\nu)| = \left|\int b(t,x,y)\left(\mu(dy)-\nu(dy)\right)\right| \leq ||b||_{bC^\alpha} d_{bC^\alpha}(\mu,\nu).$$

Indeed the bounded α-Holder distance metrizes the weak convergence, the proof will be postponed to the Appendix as theorem 9. Thus since as a function of t $B(t,x,\mu) \in L^\infty$ we have that $B^\mu \in L^\infty([0,T], bC^\alpha)$. The same is true for Σ^μ. For this reason due to Theorem 18.2.3 and 18.2.6 of [4] we have weak well posedness of the "linearized" SDE (with solution X^μ), existance of a fundamental solution and uniform gaussian estimates. In particular $p^\mu(s,x;t,y)$ will be the fundamental solution of

$$(\partial_s + \mathcal{A}_s^\mu)p^\mu(s,x;t,y) = 0,$$
$$(\partial_t - (\mathcal{A}_t^\mu)^*)p^\mu(s,x;t,y) = 0,$$

where $\mathcal{A}_s^\mu$ is the infinitesimal generator of the "linearized" SDE:

$$\mathcal{A}_s^\mu = \frac{1}{2}\sum_{i,j=1}^N c_{i,j}^\mu(t,x)\partial_{x_i x_j} + \sum_{i=1}^N B_i^\mu(t,x)\partial_{x_i},$$
$$\text{with} \qquad (c_{i,j}^\mu)_{i,j=1,\cdots N} = \Sigma^*\Sigma.$$

At this point we may define respectively the forward and the backward translation operators via $(\mu_t)_{t\in[0,T]}$:

$$U_\mu^{t,s}u(y) = \int p^\mu(s,x;t,y)u(dx), \qquad u \in \mathcal{P}(\mathbb{R}^N),$$
$$V_\mu^{s,t}u(x) = \int p^\mu(s,x;t,y)u(dy), \qquad u \in \mathcal{P}(\mathbb{R}^N),$$

and observe that the distribution of X_t^μ is absolutely continuous with density

$$u_t^\mu(y) = U_\mu^{t,0}\mu_0.$$

Thus we may define the application $\mathcal{L} : C([0,T], \mathcal{P}(\mathbb{R}^N)) \to C([0,T], \mathcal{P}(\mathbb{R}^N))$ such that $\mathcal{L}((\mu_t)_{t\in[0,T]}) = (\mathcal{L}_t^\mu)_{t\in[0,T]} = ([X_t^\mu])_{t\in[0,T]}$. Less abstractly this is the application that given a flow of marginals returns the flow of marginals of the solution of the "linearized" SDE with the first flow of marginals and initial datum μ_0.

Remark 8. *Since $\mathcal{L}_0^\mu = [X_0^\mu] = \mu_0$ for any flow of marginals the image through $\mathcal{L}$ has initial law equal to μ_0.*

Now we need to observe carefully the definition of $d_{bC^\alpha}([X_t^\mu], [X_t^\nu])$: if we remove the sup we get

$$I(f) = \int f(x)\left([X_t^\mu](dx) - [X_t^\nu](dx)\right) = \int f(x)\left(u_t^\mu(x) - u_t^\nu(x)\right)dx,$$

which may be seen as an indicator of closeness between the two distributions. This is useful because by unraveling the definition of the densities we are able to exchange the order of integration to use the backward translation operators instead:

$$\begin{aligned} I(f) &= \int\int f(x)(p^\mu - p^\nu)(0,y;t,x)\mu_0(dy)dx = \int\int f(x)(p^\mu - p^\nu)(0,y;t,x)dx\ \mu_0(dy) \\ &= \int\int\int_0^t \frac{d}{ds}\left(\int p^\mu(0,y;s,z)p^\nu(s,z;t,x)f(x)dz\right) ds\ dx\ \mu_0(dy), \end{aligned}$$

Since the fundamental solutions are $C^{1,2}$ in the time and space variables locally around (s,z) we may exchange the integral and the derivative

$$= \int\int\int_0^t \int \partial_{t_2}p^\mu(0,y;s,z)p^\nu(s,z;t,x)f(x)dz + \int p^\mu(0,y;s,z)\partial_{t_1}p^\nu(s,z;t,x)f(x)dz\ ds\ dx\ \mu_0(dy),$$

by the fact that p is the fundamental solution for both the forward and backward PDEs

$$\begin{aligned} &= \int\int\int_0^t \int(\mathcal{A}_s^\mu)^*p^\mu(0,y;s,z)p^\nu(s,z;t,x)f(x)dz - \int p^\mu(0,y;s,z)\mathcal{A}_s^\nu p^\nu(s,z;t,x)f(x)dz\ ds\ dx\ \mu_0(dy) \\ &= \int\int_0^t \int(\mathcal{A}_s^\mu)^*p^\mu(0,y;s,z)V_\nu^{s,t}f(z)dz - \int(\mathcal{A}_s^\nu)^*p^\mu(0,y;s,z)V_\nu^{s,t}f(z)dz\ ds\ \mu_0(dy) \\ &= \int\int_0^t \int p^\mu(0,y;s,z)\left(\mathcal{A}_s^\mu - \mathcal{A}_s^\nu\right)V_\nu^{s,t}f(z)dz\ ds\ \mu_0(dy) \\ &= \int\int_0^t V_\mu^{0,s}\left(\mathcal{A}_s^\mu - \mathcal{A}_s^\nu\right)V_\nu^{s,t}f(y)ds\ \mu_0(dy) = \int_0^t\int V_\mu^{0,s}\left(\mathcal{A}_s^\mu - \mathcal{A}_s^\nu\right)V_\nu^{s,t}f(y)\mu_0(dy)ds. \end{aligned}$$

Make rigorous: Which in some sense means that instead of transporting forward the initial law via different flows and testing against f we are transporting backward the test function and then testing the difference against the initial law.

At this point it's easy to see that since μ_0 is a probability distribution

$$I(f) = \int_0^t \int V_\mu^{0,s} \left(\mathcal{A}_s^\mu - \mathcal{A}_s^\nu\right) V_\nu^{s,t} f(y)\mu_0(dy)ds \le \int_0^t ||V_\mu^{0,s} \left(\mathcal{A}_s^\mu - \mathcal{A}_s^\nu\right) V_\nu^{s,t} f||_{L^\infty} ds,$$

due to the definition of $V_\mu^{0,s}$ and the fact that $p^\mu(0,x;s,y)dy$ is a probability measure we have

$$\begin{aligned}
&\le \int_0^t ||\left(\mathcal{A}_s^\mu - \mathcal{A}_s^\nu\right) V_\nu^{s,t} f||_{L^\infty} ds \\
&\le \sup_{s,x} \left(|B^\mu(s,x) - B^\nu(s,x)| + |c^\mu(s,x) - c^\nu(s,x)|\right) \int_0^t || \bigtriangledown V_\nu^{s,t} f||_{L^\infty} + ||Hess\ V_\nu^{s,t} f||_{L^\infty} ds.
\end{aligned}$$

Now by Proposition 10 if $f \in bC^\alpha$ with $||f||_{bC^\alpha} \le 1$ we have

$$|| \bigtriangledown V_\nu^{s,t} f||_{L^\infty} \le \frac{C}{\sqrt{t-s}}, \qquad ||Hess\ V_\nu^{s,t} f||_{L^\infty} \le \frac{C}{(t-s)^{1-\frac{\alpha}{2}}},$$

which yields for $I(f)$

$$\begin{aligned}
I(f) &\le C \sup_{s,x} \left(|B^\mu(s,x) - B^\nu(s,x)| + |c^\mu(s,x) - c^\nu(s,x)|\right) \int_0^t \left(\frac{1}{|t-s|^{1-\frac{\alpha}{2}}} + \frac{1}{\sqrt{t-s}}\right) ds \\
&\le C_T |t|^{\frac{\alpha}{2}} \sup_{s,x} \left(|B^\mu(s,x) - B^\nu(s,x)| + |c^\mu(s,x) - c^\nu(s,x)|\right).
\end{aligned}$$

Now we observe that since the coefficients are uniformly bC^α we have

$$|B^\mu(s,x) - B^\nu(s,x)| = \left| \int b(s,x,y)\mu_s(dy) - \int b(s,x,y)\nu_s(dy) \right| \le C d_{bC^\alpha}(\mu_s, \nu_s),$$

where $C = ||b||_{C_B^\alpha}$, a priori it depends on (s,x) but since b uniformly bC^α it can be taken uniformly in (s,x). It is also possible to prove that

$$|c^\mu(s,x) - c^\nu(s,x)| \le C||\sigma||_\infty d_{bC^\alpha}(\mu_s, \nu_s).$$

with these we can conclude that

$$I(f) \le C|t|^{\frac{\alpha}{2}} \sup_{s\in[0,t]} d_{C_B^\alpha}(\mu_s, \nu_s),$$

for any $f \in bC^\alpha$ with $||f||_{bC^\alpha} \le 1$ and thus

$$\sup_{t\in[0,T]} d_{C_B^\alpha}(\mathcal{L}_t^\mu, \mathcal{L}_t^\nu) = \sup_{t\in[0,T]} \sup_{||f||_{bC^\alpha}\le 1} I(f) \le C|T|^{\frac{\alpha}{2}} \sup_{t\in[0,T]} d_{bC^\alpha}(\mu_t, \nu_t).$$

which proves contraction for small valus of T for $\mathcal{L}$, now by a classical sequencial argument we may conclude that there exists a unique flow of marginals $(\mu_t)_{t\in[0,T]}$ such that $[X_t^\mu] = \mu_t,\ \forall t \in [0,T]$ and thus X^μ is the unique weak solution to the MKV SDE. □

5 Appendix

Theorem 9. *The bounded α-Holder distance metrizes weak convergence of measures. More precisely given $(\mu_n)_{n\in\mathbb{N}}$ and μ probability measures*

$$d_{bC^\alpha}(\mu_n, \mu) \to 0 \Leftrightarrow \mu_n \xrightarrow{d} \mu.$$

Proof. The proof will be divided in two steps and is mostly taken from `https://sites.stat.washington.edu/jaw/COURSES/520s/522/HO.522.20/ch11c.pdf`

1) First we prove that

$$\mu_n \xrightarrow{d} \mu \Leftrightarrow \int f d\mu_n \to \int f d\mu,\ \forall f \in bC^\alpha.$$

If $\mu_n \xrightarrow{d} \mu$ then equivalently $\int f\mu_n \to \int f\mu$ for any function $f \in bC$ which in particular means that it is true for any $f \in bC^\alpha$. The converse is true because if $\int f\mu_n \to \int f\mu$ for any function $f \in bC^\alpha$ then in particular it is true for any $f \in bLip$ which by Portmanteau's theorem implies weak convergence.

2) We will now prove that

$$\int f d\mu_n \to \int f d\mu \ \forall f \in bC^\alpha \Leftrightarrow d_{bC^\alpha}(\mu_n, \mu) \to 0.$$

The easy implication is the right-to-left one: indeed by comparison theorem

$$\lim_n \int f(x)\,(\mu_n(dx) - \mu(dx)) \le ||f||_{bC^\alpha} \lim_n \sup_{||g||_{bC^\alpha} \le 1} \left| \int g(x)\,(\mu_n(dx) - \mu(dx)) \right| = ||f||_{bC^\alpha} \lim_n d_{bC^\alpha}(\mu_n, \mu) \to 0.$$

The other way is more challenging, first by continuity from below of probability measures for any fixed $\epsilon > 0$ there exists K a compact set such that $\mu(K) > 1 - \epsilon$. Let $\mathcal{H} = \{f \in bC^\alpha \mid ||f||_{bC^\alpha} \le 1\}$, if we restrict each of these functions on K we have that $\mathcal{H}|_K$ is totally bounded with respect to the $||\cdot||_\infty$ norm by Ascoli-Arzelà's theorem, in particular $\exists k$ finite and $f_1, \cdots f_k \in \mathcal{H}|_K$ such that for any $f \in \mathcal{H}$ $\exists j$ such that $\sup_K |f - f_j| \le \epsilon$.

Now if we consider $d(x,y) = |x - y|$ and $K^\epsilon = \left\{x \in \mathbb{R}^N \mid d(x,K) \le \epsilon\right\}$ and f, f_j as before we have

$$\sup_{x \in K^\epsilon} |f(x) - f_j(x)| \le \sup_{x \in K^\epsilon} (|f(x) - f(y_x)| + |f(y_x) - f_j(y_x)| + |f_j(y_x) - f_j(x)|) \le \sup_{x \in K^\epsilon} (2\epsilon^\alpha + \epsilon) \le C_\alpha \epsilon^\alpha.$$

where y_x is a point in K such that $d(x, y_x) < \epsilon$. C_α may be taken uniformly of ϵ as long as $\epsilon \le 1$.

Let $g(x) = \max\left(0, 1 - \frac{d(x,K)}{\epsilon}\right)$, evidently $g \in bLip \subseteq bC^\alpha$ and $\mathbb{1}_K \le g \le \mathbb{1}_{K^\epsilon}$. Thus by taking n big enough we have by convergence against bC^α functions that

$$\mu_n(K^\epsilon) \ge \int g(x)\mu_n(dx) > 1 - 2\epsilon.$$

Thus by taking $f \in \mathcal{H}$ and the associated f_j we have

$$\begin{aligned}
\left| \int f(x)\,(\mu_n(dx) - \mu(dx)) \right| &= \left| \int (f(x) - f_j(x))\,(\mu_n(dx) - \mu(dx)) \right| + \left| \int f_j(x)\,(\mu_n(dx) - \mu(dx)) \right| \\
&\le \left| \int (f(x) - f_j(x))\mu_n(dx) \right| + \left| \int (f(x) - f_j(x))\mu(dx) \right| + \left| \int f_j(x)\,(\mu_n(dx) - \mu(dx)) \right| \\
&\le \left| \int_{K^\epsilon} (f(x) - f_j(x))\mu_n(dx) \right| + \left| \int_{(K^\epsilon)^c} (f(x) - f_j(x))\mu_n(dx) \right| + \\
&\quad + \left| \int_{K^\epsilon} (f(x) - f_j(x))\mu(dx) \right| + \left| \int_{(K^\epsilon)^c} (f(x) - f_j(x))\mu(dx) \right| + \\
&\quad + \left| \int f_j(x)\,(\mu_n(dx) - \mu(dx)) \right| \\
&\le C_\alpha \epsilon^\alpha + 4\epsilon + C_\alpha \epsilon^\alpha + 2\epsilon + \epsilon \le C_\alpha \epsilon^\alpha,
\end{aligned}$$

where the last term gets bounded by taking n big enough and by using convergence against bC^α functions, this gives us the final result. □

Proposition 10. *If f is a bC^α function with $||f||_{bC^\alpha} \le 1$ then*

1. $|| \bigtriangledown\ V_\nu^{s,t} f||_{L^\infty} \le \frac{C}{\sqrt{t-s}}$,
2. $||Hess\ V_\nu^{s,t} f||_{L^\infty} \le \frac{C}{|t-s|^{1-\frac{\alpha}{2}}}$.

Proof. 1. Simply by gaussian estimates (Theorem 20.2.5 of [4])

$$\begin{aligned}
\left|\partial_{x_i} V_\nu^{s,t} f(x)\right| &= \left| \int \partial_{x_i} p^\nu(s, x; t, y) f(y) dy \right| \\
&\le \frac{C}{\sqrt{t-s}} \int \Gamma^+(t-s, x-y)|f(y)|dy \le \frac{C}{\sqrt{t-s}}
\end{aligned}$$

2. For the second derivates it is imperative the use of the α-Holderian structure of f

$$
\begin{aligned}
\left|\partial_{x_i x_j} V_\nu^{s,t} f(x)\right| &= \left|\int \partial_{x_i x_j} p^\nu(s,x;t,y) f(y) dy\right| \\
&\leq \left|\int \partial_{x_i x_j} p^\nu(s,x;t,y)(f(y)-f(x)) dy\right| + \left|\int \partial_{x_i x_j} p^\nu(s,x;t,y) dy f(x)\right| \\
&\leq \int |\partial_{x_i x_j} p^\nu(s,x;t,y)||x-y|^\alpha dy + \left|\partial_{x_i x_j} \underbrace{\int p^\nu(s,x;t,y) dy}_{=1}\right| |f(x)| \\
&\leq \frac{C}{|t-s|} \int \Gamma^+(t-s, x-y)|x-y|^\alpha dy + 0 \leq \frac{C_{B,\alpha}}{|t-s|^{1-\frac{\alpha}{2}}}.
\end{aligned}
$$

by Theorem 20.2.5 and Lemma 20.3.4 of [4].

□

References

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